Chapter 2 Second-Order Differential Equations

2.1 Preliminary Concepts

常微分方程式之階數為 2 或者 2 以上，為高階常微分方程式

$$F(x, y, y', y'') = 0$$

$$y'' = x^3$$

$$xy'' - \cos(y) = e^x$$

$$y'' - 4xy' + y = 2$$

$y(x) = 6\cos(4x) - 17\sin(4x)$ is a solution of

$y'' + 16y = 0$ for all real x

$y(x) = x^3\cos(\ln(x))$ is a solution of

$x^2y'' - 5xy' + 10y = 0$ for $x > 0$

$$R(x)y'' + P(x)y' + Q(x)y = S(x)$$

$$y'' + p(x)y' + q(x)y = f(x)$$

2.2 Theory of Solutions of $y'' + p(x)y' + q(x)y = f(x)$

$$y'' - 12x = 0$$

$$y'' = 12x$$

$$y' = \int y''(x)dx = \int 12x dx = 6x^2 + C$$

$$y(x) = \int y'(x)dx = \int (6x^2 + C)dx = 2x^3 + Cx + K$$

For any choice of C and K, we can graph the integral curves

$y = 2x^3 + Cx + K$ as curves in the plane

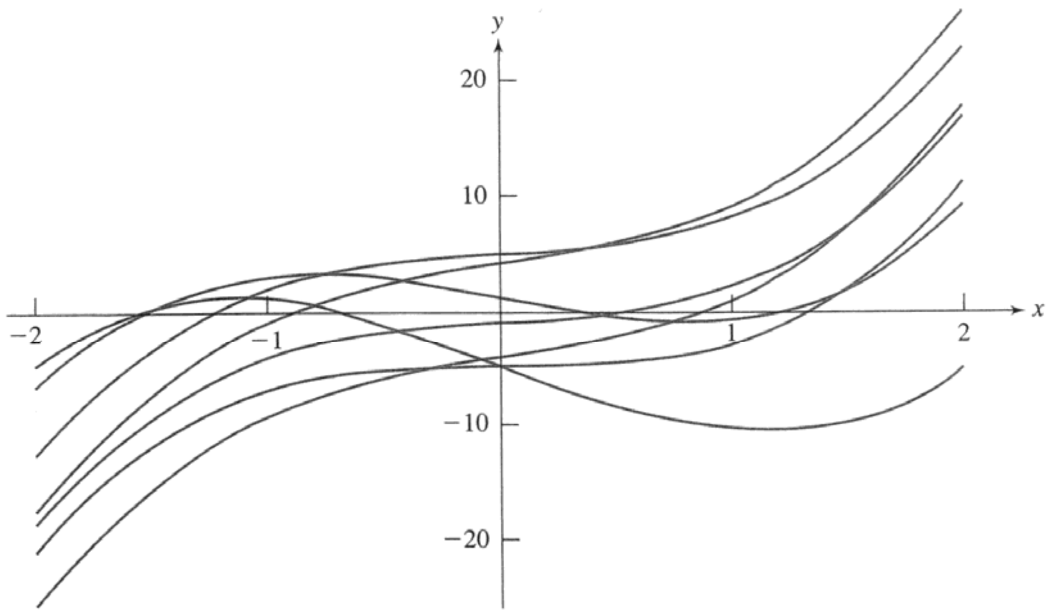


FIGURE 2.1 *Graphs of* $y = 2x^3 + Cx + K$ *for various values of* C *and* K.

If initial conditions $y(0) = 3$ ， $y'(0) = -1$

$y(0) = K = 3$

$y(x) = 2x^3 + Cx + 3$

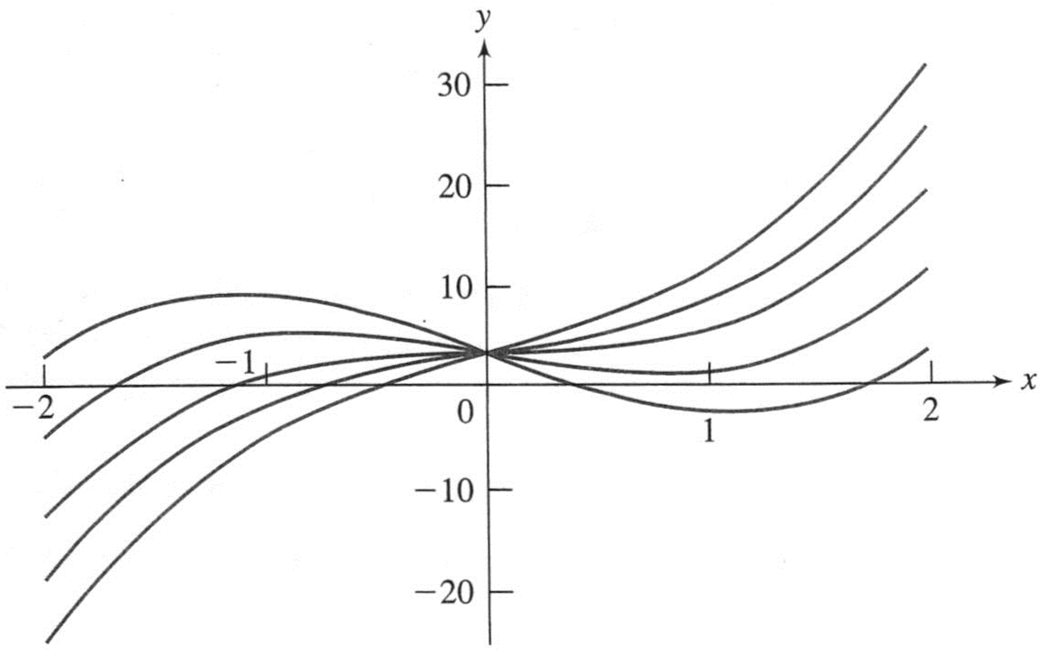


FIGURE 2.2 *Graphs of* $y = 2x^3 + Cx + 3$ *for various values of* C.

Since $y'(x) = 6x^2 + C$, this requites that $C = -1$.

$$y(x) = 2x^3 - x + 3$$

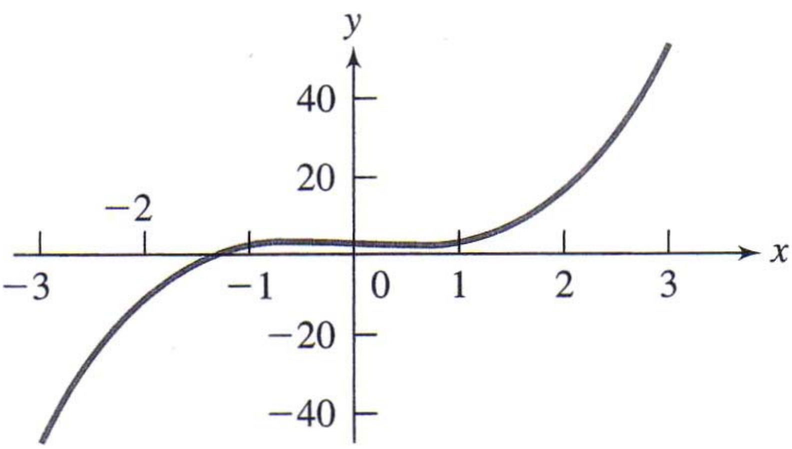


FIGURE 2.3 *Graph of* $y = 2x^3 - x + 3$.

The curve passes (0,3) and has slope -1 at this point

2.2.1 The Homogeneous Equation $y''+p(x)y'+q(x)y=0$

- $y''+p(x)y'+q(x)y=f(x)$; $y(x_0)=A, y'(x_0)=B$

  When $f(x)$ is zero, the resulting equation $y''+p(x)y'+q(x)y=0$ is called *homogeneous*

- A *linear combination* of solutions $y_1(x)$ and $y_2(x)$

  $c_1y_1(x)+c_2y_2(x)$

(Thm 2.2)

Let $y_1(x)$ and $y_2(x)$ be solutions of $y''+p(x)y'+q(x)y=0$ on an interval I. Then any linear combination of these solutions is also a solution.

[Proof]: Let c_1 and c_2 be real numbers. Substituting $y(x)=c_1y_1(x)+c_2y_2(x)$ into the differential equation, we obtain

$$\begin{aligned}
&(c_1y_1+c_2y_2)''+p(x)(c_1y_1+c_2y_2)'+q(x)(c_1y_1+c_2y_2)\\
&=c_1y_1''+c_2y_2''+c_1p(x)y_1'+c_2p(x)y_2'+c_1q(x)y_1+c_2q(x)y_2\\
&=c_1\left[y_1''+p(x)y_1'+q(x)y_1\right]+c_2\left[y_2''+p(x)y_2'+q(x)y_2\right]\\
&=0+0=0
\end{aligned}$$

because of the assumption that y_1 and y_2 are both solutions.

(Def 2.1) Linear Dependence, Independence

Two functions f and g are linearly dependent on an open interval I if, for some constant c, $f(x) = cg(x)$ for all x in I. If f and g are not linearly dependent on an open interval I, then they are said to be linearly independent on the interval

- A simple test to tell whether two solutions of equation are linearly independent

  Define the *Wronskian* of solutions y_1 and y_2 to be

$$W(x) = y_1(x)y_2'(x) - y_1'y_2(x).$$

  This is the 2×2 determinant

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}$$

Ex27:

$$W(x) = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix}$$

$$= \cos^2(x) + \sin^2(x) = 1 \neq 0.$$

(Thm 2.4)

Let y_1 and y_2 be linearly independent solutions of $y''+p(x)y'+q(x)y=0$ on an open interval I. Then, every solution of this differential equation on I is a linear combination of y_1 and y_2

◆ $y''+p(x)y'+q(x)y=0;\ \ y(x_0)=A,\ \ y'(x_0)=B.$

$y_1(x_0)c_1+y_2(x_0)c_2=A$

$y_1'(x_0)c_1+y_2'(x_0)c_2=B$

Cramer's Rule

$$c_1=\frac{Ay_2'(x_0)-By_2(x_0)}{W(x_0)},\ \ c_2=\frac{By_1(x_0)-Ay_1'(x_0)}{W(x_0)}$$

2.2.2 The Nonhomogeneous Equation $y''+p(x)y'+q(x)y=f(x)$

(Thm 2.5)

y_1 and y_2 be a fundamental set of solutions of $y''+p(x)y'+q(x)y=0$ on an open interval I. Let y_p be any solution of equation $y''+p(x)y'+q(x)y=f(x)$. Then, for any solution φ of equation $y''+p(x)y'+q(x)y=f(x)$, there exist numbers c_1 and c_2 such that $\varphi=c_1y_1+c_2y_2+y_p$

[Proof]: Since φ and y_p are both solutions of equation

$$y''+p(x)y'+q(x)y=f(x)$$

$$\varphi''+p\varphi'+q\varphi-\left(y_p''+py_p'+qy_p\right)=f-f=0$$

$$\left(\varphi-y_p\right)''+p\left(\varphi-y_p\right)'+q\left(\varphi-y_p\right)=0$$

$\varphi-y_p$ is a solution of $y''+py'+qy=0$.

$$\varphi-y_p=c_1y_1+c_2y_2,\ \ \varphi=c_1y_1+c_2y_2+y_p$$

Exercise L :

In each problem, (a) verify that y_1 and y_2 are solutions of the differential equation, (b) show that their Wronskian is not zero, (c) write the general solution of the differential equation, and (d) find the solution of the initial value problem.

1. $y''+9y=0; y(\pi/3)=0, y'(\pi/3)=1$

$y_1(x)=\cos(3x), y_2(x)=\sin(3x)$

{(b) $W=\begin{vmatrix} \cos(3x) & \sin(3x) \\ -3\sin(3x) & 3\cos(3x) \end{vmatrix}=3$

(c) $y=c_1\cos(3x)+c_2\sin(3x)$

(d) $y=-\frac{1}{3}\sin(3x)$ }

2. $y''+11y'+24y=0; y(0)=1, y'(0)=4$

$y_1(x)=e^{-3x}, y_2(x)=e^{-8x}$

{(b) $W=\begin{vmatrix} e^{-3x} & e^{-8x} \\ -3e^{-3x} & -8e^{-8x} \end{vmatrix}=-5e^{-11x}$

(c) $y=c_1e^{-3x}+c_2e^{-8x}$

(d) $y=\frac{12}{5}e^{-3x}-\frac{7}{5}e^{-8x}$ }

◆ 高階線性常微分方程式

$$y^{(n)}+a_{n-1}(x)y^{(n-1)}+....+a_1(x)y'+a_0(x)y=R(x)$$

可根據其$R(x)$是否出現分成兩類

$R(x)$不出現:稱其為*齊性方程式*(homogeneous equation)

$R(x)$出現:稱其為*非齊性方程式*(nonhomogeneous equation)

◆ 求解壹 n 階線性非齊性 O.D.E.

$y^{(n)}+a_{n-1}(x)y^{(n-1)}+....+a_1(x)y'+a_0(x)y=R(x)$有三步驟

(1) 求 Homogeneous solution y_h:

$y^{(n)}+a_{n-1}(x)y^{(n-1)}+....+a_1(x)y'+a_0(x)y=0$之通解

$y_h=c_1y_1+....+c_ny_n$

(其中$W(y_1,\cdots,y_n)\neq 0$)

(2) 求 Particular solution y_p:

$y^{(n)}+a_{n-1}(x)y^{(n-1)}+....+a_1(x)y'+a_0(x)y=R(x)$

之任一個特解y_p

(3) 原 O.D.E.之通解為$y=y_h+y_p$

常見高階線性 O.D.E. 可分為 3 類:

（1） 高階線性常係數 O.D.E.

$$a_n y^{(n)}+....+a_1 y'+a_0 y=R(x)$$

（2） 高階等維線性 O.D.E.

$$a_n x^n y^{(n)}+....+a_1 xy'+a_0 y=R(x)$$

（3） 高階一般變係數之線性 O.D.E.

$$a_n(x)y^{(n)}+....+a_1(x)y'+a_0(x)y=R(x)$$

2.4 The Constant Coefficient Homogeneous Linear Equation

◆ 常係數線性常微分方程式：

首先考慮$n=2$時，即求解$y''+a_1y'+a_0y=0$之通解

1. 設解為$y=e^{mx}$，則得$y'=me^{mx}$，$y''=m^2e^{mx}$

2. 將y，y'，y''代入得：$m^2+a_1m+a_0=0$（稱為 characteristic equation, 特徵方程式）

3. （1）. $a_1^2-4a_0>0$時，可得兩相異實根m_1，m_2

$y_1=e^{m_1x}$，$y_2=e^{m_2x}$均為解

$W\left(e^{m_1x},e^{m_2x}\right)\neq 0$

$y=c_1e^{m_1x}+c_2e^{m_2x}$

（2）. $a_1{}^2 - 4a_0 = 0$，兩實重根$m_1 = m_2$

$y_1 = e^{m_1 x}$

要找到兩個滿足線性獨立之解

故此時，缺少一個線性獨立解

可令$y_2 = xe^{m_1 x}$

因$W\left(e^{m_1 x}, xe^{m_1 x}\right) \neq 0$

$y = \left(c_1 + c_2 x\right) e^{m_1 x}$

（3）. $a_1{}^2 - 4a_0 < 0$時，兩共軛複根

$m_{1,} m_2 = \alpha \pm i\beta$

$y_1 = e^{(\alpha + i\beta)x}$ ， $y_2 = e^{(\alpha - i\beta)x}$均為其解

$W\left(e^{(\alpha + i\beta)x}, e^{(\alpha - i\beta)x}\right) \neq 0$

$y = c_1 e^{(\alpha + i\beta)x} + c_2 e^{(\alpha - i\beta)x}$

Ex28：求下列 O.D.E 之通解： $y'''+4y''-3y'-18y=0$

［解］:

◆ 求任意特解 $y_p(x)$ ，方法有兩種:

(1) 待定係數法

(2) 參數變易法

◆ 待定係數法僅適用於常係數線性 O.D.E

待定係數法限制 $R(x)$ 之型式如下：

A（常數），e^{ax} ，$\cos bx$ ，$\sin bx$ ，x 之多項式

(表二)

$R(x)$	$y_p(x)$之假設型式
k	A
e^{ax}	Ae^{ax}
$\cos ax$	$A\cos ax+B\sin ax$
$\sin ax$	$A\cos ax+B\sin ax$
$a_nx^n+\ldots+a_1x+a_0$	$A_nx^n+A_{n\text{-}1}x^{n-1}+\ldots+A_1x+A_0$
$e^{ax}\cos bx$ or $e^{ax}\sin bx$	$e^{ax}(A\cos bx+B\sin bx)$
$e^{ax}(a_nx^n+\ldots+a_1x+a_0)$	$e^{ax}(A_nx^n+\ldots+A_1x+A_0)$
$\cos bx(a_nx^n+\ldots+a_1x+a_0)$	$(A_nx^n+\ldots+A_1x+A_0)\cos bx$ $+(B_nx^n+\ldots+B_1x+B_0)\sin bx$
$e^{ax}\cos bx(a_nx^n+\ldots+a_1x+a_0)$	$e^{ax}(A_nx^n+\ldots+A_1x+A_0)\cos bx$ $+e^{ax}(B_nx^n+\ldots+B_1x+B_0)\sin bx$

Ex29：求下列 O.D.E 之通解： $y''-y'-2y=e^{3x}$

[解]:

Ex30：求下列 O.D.E 之通解： $y''+2y'-3y=4\sin(2t)$

［解］:

Ex31：求解下列方程式之通解：$y''-5y'+6y=e^{3x}$

[解]:1.設齊性解 $y_h \equiv e^{mx}$，代入得

特徵方程式：$m^2-5m+6=0$, $\therefore m=2,3$

故 $y_h=c_1e^{2x}+c_2e^{3x}$

2.利用(表二)，可令 $y_p=Ae^{3x}$

得 $y_p'=3Ae^{3x}$ ，$y_p''=9Ae^{3x}$

則代入原式得：

$(9A-5(3A)+6A)e^{3x}=e^{3x}$

$\therefore 0\,e^{3x}=e^{3x}$ (矛盾)

故(表二)之假設方式不適用於本題之 $y_p(x)$ 的求解

◆ 當 $y_p(x)$ 之假設項與 $y_h(x)$ 所含 n 個線性獨立齊性解產生重複時，我們必須做修正

結論:

當使用待定係數法求 y_p 時，y_p 之假設項中與 y_h 產生重複,則可乘以(x^m) (m 為正整數)於 y_p 之假設項中與 y_h 產生重複者，其中 m 為使其不產生重複之最小正整數

Ex32：求解下列方程式之通解： $y''-5y'+6y=e^{3x}$

［解］:

Ex33：求下列 O.D.E 之通解： $y''-2y'+y=e^{x}$

［解］:

Ex34：求下列 O.D.E 之通解： $y''' + y' = \sin x$

［解］:

參數變易法 (Variation Parameter Method)

利用此法求 $y^{(n)}+a_{n-1}(x)y^{(n-1)}+...+a_1(x)y'+a_0(x)y=R(x)$

之特解 y_p，其主要先決條件為原 O.D.E 之相應齊性 O.D.E.

的 n 個線性獨立解已經求得。參數變易法適用於求任何 n

階線性 O.D.E

◆ $y''+a_1(x)y'+a_0(x)y=R(x)$

$y_h=c_1y_1(x)+c_2y_2(x)$

設 $y_p(x)=\phi_1(x)y_1(x)+\phi_2(x)y_2(x)$

得 $y_p'=\phi_1y_1'+\phi_2y_2'+\phi_1'y_1+\phi_2'y_2$

若令 $\phi_1'y_1+\phi_2'y_2=0$ ……………….….(1)

可得 $y_p'=\phi_1y_1'+\phi_2y_2'$

且 $y_p''=\phi_1y_1''+\phi_2y_2''+\phi_1'y_1'+\phi_2'y_2'$

將 y_p,y_p',y_p'' 代入 $y''+a_1(x)y'+a_0(x)y=R(x)$，可得

$(\phi_1y_1''+\phi_2y_2''+\phi_1'y_1'+\phi_2'y_2')+a_1(\phi_1y_1'+\phi_2y_2')+a_0(\phi_1y_1+\phi_2y_2)=R(x)$

已知 y_1,y_2 分別為 $y''+a_1y'+a_0y=0$ 之解

$\phi_1(y_1''+a_1y_1'+a_0y_1)+\phi_2(y_2''+a_1y_2'+a_0y_2)+\phi_1'y_1'+\phi_2'y_2'=R(x)$

$\phi_1'y_1'+\phi_2'y_2'=R(x)$ …………………(2)

聯立(1)與(2)得

$$\phi_1' = \frac{-R(x)y_2}{W(y_1, y_2)} \quad 且 \quad \phi_2' = \frac{+R(x)y_1}{W(y_1, y_2)}$$

$$\phi_1 = \int \frac{-R(x)y_2}{W(y_1, y_2)}dx \quad 且 \quad \phi_2 = \int \frac{R(x)y_1}{W(y_1, y_2)}dx$$

$$y_p(x) = y_1(x)\int \frac{-R(x)y_2}{W(y_1, y_2)}dx + y_2(x)\int \frac{R(x)y_1}{W(y_1, y_2)}dx$$

其中$W(y_1, y_2)$為 Wronskian 行列式

Ex35：Find the general solution for $y'' - 2y' + y = \frac{e^x}{1-x}$

[解]:

Exercise M:

1. $y''-2y'+10y=0$ $\{y=e^{x}\left[c_1\cos 3x+c_2\sin 3x\right]\}$

2. $y''+10y'+26y=0$ $\{y=e^{-5x}\left[c_1\cos x+c_2\sin x\right]\}$

3. $\frac{d^3y}{dx^3}-\frac{d^2y}{dx^2}=2\sin x-5e^{x}$

$\{y=c_1+c_2x+c_3e^{x}+\cos x+\sin x-5xe^{x}\}$

4. $y'''-2y''-y'+2y=0; y(0)=3,\ \ y'(0)=0,\ \ y''(0)=3$

$\{y=\frac{3}{2}e^{x}+\frac{3}{2}e^{-x}\}$

5. $y''-y'-2y=10\cos x$ $\{y=c_1e^{-x}+c_2e^{2x}-3\cos x-\sin x\}$

6. $y''+2y'+y=xe^{-x}$ $\{y=\left(c_1+c_2x\right)e^{-x}+\frac{1}{6}x^3e^{-x}\}$

7. $y''-2y'+2y=\cos x$

$\{y=e^{x}\left(c_1\cos x+c_2\sin x\right)+\frac{1}{5}\left[\cos x-2\sin x\right]\}$

8. $y''-4y=-7e^{2x}+x; y(0)=1, y'(0)=3$

$\{y=\frac{7}{4}e^{2x}-\frac{3}{4}e^{-2x}-\frac{7}{4}xe^{2x}-\frac{1}{4}x\}$

等維線性常微分方程式

(1) Cauchy 等維線性 O.D.E.

(2) Legendre 等維線性 O.D.E

2.5 Euler’s Equation

◆ Cauchy 等維線性 O.D.E.

$a_n x^n y^{(n)} + ... + a_1 xy' + a_0 y = R(x)$，此為 n 階 Cauchy 等維線性常微分方程式，$(a_n \neq 0)$。利用變數變換將其化為常係數線性 O.D.E.

解法如下:

(1) 令 $x = e^t$，或 $t = \ell nx$，且 $D_t \equiv \dfrac{d}{dt}$

其中 $x^n \dfrac{d^n y}{dx^n} = x^n D^n y = D_t (D_t - 1)(D_t - 2)...(D_t - n + 1) y$

(2) 利用常係數線性 O.D.E.之方法來求解

(3) $t = \ell nx$

Ex36：Find a general solution for $x^2 y'' + xy' - y = x^3 e^x$

［解］:

Ex37：Find the general solution for $x^2 y'' - 2xy' + 2y = x^3 \cos x$

［解］:

Legendre 等維線性 O.D.E

$a_n(bx+c)^n y^{(n)} + ... + a_1(bx+c)y' + a_0 y = R(x)$ 為 n 階 Legendre 等維線性常微分方程式，利用變數變換技巧轉成 Cauchy 等維線性 O.D.E.

Ex38：求解下列 O.D.E.之通解：

$$(3x+2)^2 y'' + 3(3x+2)y' - 36y = 3x^2 + 4x + 1$$

［解］:

Exercise N:

1. $x^2 y'' + 2xy' - 6y = 0$ $\{ y = c_1 x^2 + c_2 x^{-3} \}$

2. $x^2 y'' + xy' + 4y = 0$ $\{ y = c_1 \cos\left[2\ln(x)\right] + c_2 \sin\left[2\ln(x)\right] \}$

3. $x^2 y'' - xy' = 0; y(2) = 5, y'(2) = 8$ $\{ y = -3 + 2x^2 \}$

4. $x^2 y'' - 2xy' + 2y = 6\ln x;\quad x > 0$ $\{ y = c_1 x + c_2 x^2 + 3\ln x + \dfrac{9}{2} \}$

5. $(2x+1)^2 y'' - (12x+6)y' + 16y = 2$

$\{ y = (2x+1)^2\left[c_1 + c_2 \ln|2x+1|\right] + \dfrac{1}{8} \}$

6. $(x^2 + 6x + 9)\dfrac{d^2y}{dx^2} + (3x+9)\dfrac{dy}{dx} + 2y = 0; y(0) = 0,\ \dfrac{dy}{dx}(0) = 1$

{

$$y = \frac{1}{x+3}\left[-9\sin(\ln 3)\cos(\ln|x+3|) + 9\cos(\ln 3)\sin(\ln|x+3|)\right]$$

}

2.3 Reduction of Order

◆ 可用因變數變更法求解二階變係數線性常微分方程式

$y''+P(x)y'+Q(x)y=R(x)$

(1) 觀察出齊性方程式 $y''+P(x)y'+Q(x)y=0$ 之一個解 $u(x)$

時，則令 $y(x)=u(x)v(x)$

(2) $y'=uv'+u'v$

$y''=u'v'+uv''+u''v+u'v'=2u'v'+uv''+u''v$

代入原式，可得

$uv''+2u'v'+Puv'+v(u''+Pu'+Qu)=R$

$$v''+\frac{2u'+Pu}{u}v'=\frac{R}{u}$$

可令 $t=v'$，得 $t'+\dfrac{2u'+Pu}{u}t=\dfrac{R}{u}$

此為因變數 t 之壹階線性 O.D.E.

Ex39: $y'' - (3/x)y' + (4/x^2)y = 0$ for $x > 0$, given $y_1(x) = x^2$ is one solution

[解]:

Exercise O:

1. $2xy''+(1-4x)y'+(2x-1)y=e^x$ $\{ y=e^x\left[c_1\sqrt{x}+c_2+x\right]\}$

2. $y''-10y'+25y=0; y_1(x)=e^{5x}$ $\{ y=c_1e^{5x}+c_2xe^{5x}\}$

3. $y''+\frac{2}{x}y'+y=3$ （已知 $(\frac{\sin x}{x})''+\frac{2}{x}(\frac{\sin x}{x})'+\frac{\sin x}{x}=0$）

$\{ y=c_1\frac{\sin x}{x}-c_2\frac{\cos x}{x}+3\}$

2.7 Applications

Ex40: Determine the $i(t)$ in the circuit.

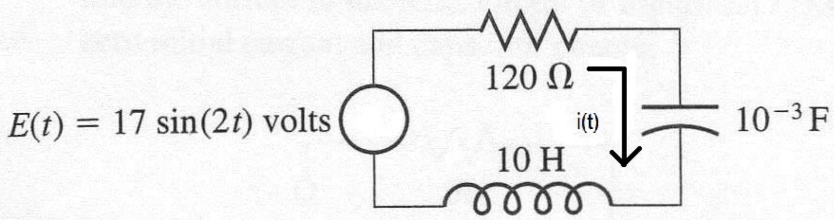


FIGURE 2.13

[解]:

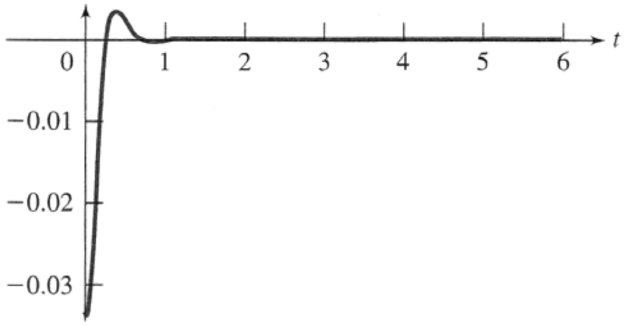


FIGURE 2.14 *Transient part of the current for the circuit of Figure 2.13.*

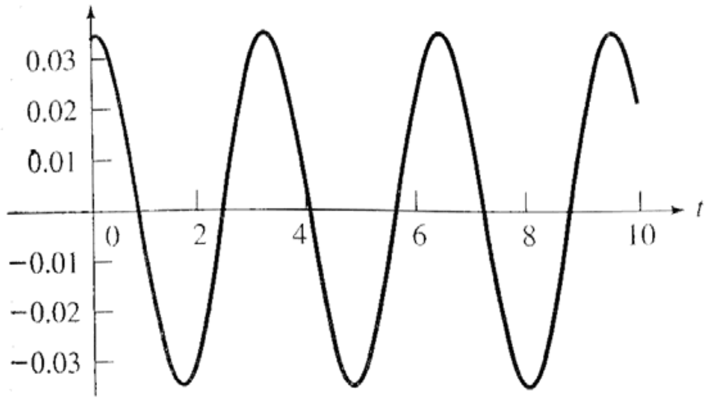
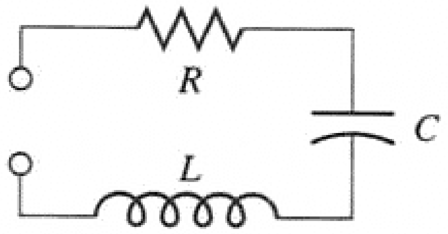


FIGURE 2.15 *Steady-state part of the current for the circuit of Figure 2.13.*

Exercise P:

1. Find the current in the following RLC circuit. Assume zero initial current and capacitor charge. ($R = 10\Omega$, $L = 0.5H$, $C = 0.02F$, $E(t) = 120\sin(20t)V$)



$$\{\, q(t) = \frac{48}{125}e^{-10t} + \frac{48}{5}te^{-10t} - \frac{36}{125}\sin 20t - \frac{48}{125}\cos 20t$$

$$i(t) = \frac{144}{25}e^{-10t} - 96te^{-10t} - \frac{144}{25}\cos 20t + \frac{192}{25}\sin 20t \,\}$$